# Lecture 12

N.E.S.M

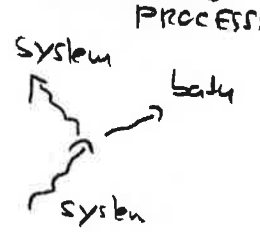
## Applying the Quantum Optical Master Equation

### Other forms of dissipation (and jump operators)

Energy relaxation is characterized by jump operators $\hat{C} = \hat{a}$ and $\hat{C}^{+} = \hat{a}^{+}$

$$\hat{H}_{int} = \sum_k \left( \hat{C} \hat{b}_k^{+} + \hat{C}^{+} \hat{b}_k \right) \hbar g_k$$

SCATTERING PROCESS:




**Dephasing interaction:**

$$\hat{H}_{int} = \sum_k \left\{ (a^{+}a) b_k^{+} + b_k (a^{+}a) \right\} \hbar g_k$$

Thus the interaction Hamiltonian destroys or creates bath quanta but leaves system quanta in tact (unchanged) $\Rightarrow$ DEPHASING PROCESS.

$$\frac{1}{T_\Phi} = 2\pi D(\omega) |g(\omega)|^2 \qquad \text{Dephasing rate}$$

This yields the Master Equation for dephasing

$$\dot{\rho} = [a^{+}a, \rho](-i\omega) + \frac{1}{2}\frac{1}{T_\Phi}(2\bar{n}_m + 1)\left\{ 2a^{+}a \rho a^{+}a - (a^{+}a)^2 \rho - \rho (a^{+}a)^2 \right\}$$

( This can be derived directly from Master equation with $\hat{C} = a^{+}a$ $\hat{C}^{+} = (a^{+}a)^{+} = a^{+}a$

Proof:

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{i}{\hbar}[H_0, \rho] + \frac{\Phi}{2}(\bar{n}_m + 1)\left( 2\hat{c}\rho\hat{c}^{+} - cc^{+}\rho - \rho c^{+}c \right) \\ &\quad + \frac{\Phi}{2}\bar{n}_m \left( 2c^{+}\rho c - c^{+}c\rho - \rho c c^{+} \right) \\ &= \frac{i}{\hbar}[H_0, \rho] + \frac{\Phi}{2}(2\bar{n}_m + 1)\left( 2c\rho c - c^2\rho - (c)^2 \right) \end{aligned}$$

} Quantum Master Equation.

Thus: a energy and phase damped oscillator obeys in the Fock basis

$$\frac{d}{dt}\rho_{nm} = i\omega(n-m) - \frac{1}{2}\left(\frac{1}{T_\Phi}\right)(2\bar{n}_m + 1)(n-m)^2 \cdot \rho_{nm}$$

DEPHASING PROCESS.

or with Energy relaxation:

$$\frac{d\rho_{nm}}{dt} = \left[ -\frac{(m+n)}{2T_1} + \frac{(m-n)^2}{T_\Phi} \right] \rho_{m,n} + \frac{\sqrt{m+1}\sqrt{n+1}}{T_1} \rho_{m+1,n+1}$$

$\frac{1}{T_1}$: energy relaxation $\qquad \frac{1}{T_\Phi}$: dephasing rate

Lecture 12

## QUANTUM MASTER EQUATION FOR A TWO LEVEL SYSTEM

$\{\sigma^+, \sigma^-, \sigma_z\}$ SYSTEM OPERATOR

$$\boxed{\begin{aligned} H_{BATH} &= \sum_k \hbar\omega_k \hat{b}_k^+ \hat{b}_k \\ H_{SR} &= \sum_k g_k \hbar\, \sigma^+ b_k + \sigma^- b_k^+ \\ H_S &= \sigma_z \cdot \frac{\hbar\omega_{ATOM}}{2} \end{aligned}}$$

$\sigma^+|g\rangle = |e\rangle$

$\sigma^-|e\rangle = |g\rangle$

$|e\rangle$, $\omega_{ATOM}$, $|g\rangle$

$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ PAULI MATRICES

$\Rightarrow$ Quantum Master Equation for radiatively damped 2-level system.

$$\boxed{\begin{aligned} \dot{\varrho} = -i\tfrac{1}{2}\omega_A[\sigma_z, \varrho] &+ \tfrac{\gamma}{2}(\bar{n}_m+1)(2\sigma_-\varrho\sigma_+ - \sigma_+\sigma_-\varrho - \varrho\sigma_+\sigma_-) \\ &+ \tfrac{\gamma}{2}\bar{n}_m(2\sigma_+\varrho\sigma_- - \sigma_-\sigma_+\varrho - \varrho\sigma_-\sigma_+) \end{aligned}}$$

Thus : $a \to \sigma_-$ and $a \to \sigma_+$ compared to damped H.O.

Equations of motion for the operators:

$Tr\{\varrho\sigma^+_z\} = \langle\sigma^+_z\rangle$

DRIVE-TERM (see iv)

$$\boxed{\begin{aligned} \frac{d}{dt}\langle\sigma^+\rangle &= -\tfrac{1}{2}\gamma(2\bar{n}+1)\langle\sigma^+\rangle \\ \frac{d}{dt}\langle\sigma^-\rangle &= -\tfrac{1}{2}\gamma(2\bar{n}+1)\langle\sigma^-\rangle \\ \frac{d}{dt}\langle\sigma_z\rangle &= -\gamma(2\bar{n}+1)\langle\sigma_z\rangle - \gamma \end{aligned}}$$

$+ i\Omega$

$-i\Omega^*$

$+\tfrac{1}{2}i(\Omega^*\langle\sigma^+\rangle - \Omega^*\langle\sigma^-\rangle)$

### COMMENTS ON THE MODEL 2 EQUATIONS

(i) Stationary solutions are $\langle\sigma^+\rangle = \langle\sigma^-\rangle = 0$ and $\langle S_z\rangle = -\frac{1}{2\bar{n}+1}$

$\bar{n} = (\exp(\hbar\omega/k_BT) - 1)^{-1}$. Note that for $T\to\infty$ $\langle S_z\rangle = 0$ i.e. population balance between state is reached $\langle S_z\rangle = -\varrho_{ee} + \varrho_{gg} = 0$ for $T\to\infty$

(ii) Comparison to the harmonic oscillator:

$\langle\dot{\sigma}_z\rangle = -\gamma(2\bar{n}+1)\langle\sigma_z\rangle$ Thus the RELAXATION is TEMPERATURE DEPENDENT OF THE POPULATION different to the harmonic oscillator ( $\frac{\partial}{\partial t}P_n = -\gamma P_n$ )

(iii) Zero Temperature bath is corresponding e.g. to the ATOMIC SPONTANEOUS EMISSION RATE. Since $\frac{\hbar\omega}{k_BT} \simeq \infty$ at optical frequencies.

$\gamma = |g_k|^2 D(\omega)\cdot 2\pi$ is the SPONT. EMISSION RATE $D(\omega)$: density of EM modes

(iv) Inclusion of a drive field to excite ATOM.

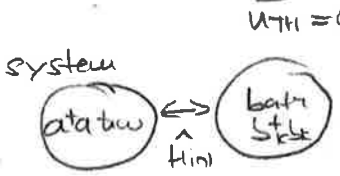
$$\boxed{H_{DRIVE} = \tfrac{1}{2}(\hbar\Omega^* a + \hbar\Omega a^+)}$$

## Quantum Reservoir Engineering

reference: Poyatos Cirac Zoller 1996

Powerful concept to create quantum states (i.e. nonclassical states) despite the interaction with a bath

Master equation for zero temp. bath:




$$\dot{\rho}_s = \kappa\{2\hat{c}\rho\hat{c}^+ - \hat{c}^+\rho\hat{c} - \hat{c}\rho\hat{c}^+\} \qquad \text{for } \bar{n}_{TH}=0$$

$\hat{c}$: jump operator    relaxation $\hat{c}=\hat{a}$

dephasing $\hat{c}=\hat{a}^+\hat{a}$

One can experimentally however also construct other interaction operators / Hamiltonians.

Concept: "POINTER BASIS" (Zurek) "The coupling to the environment singles out a preferred set of states, the pointer basis".

| The system relaxes into the Eigenstate of $\hat{c}$ with ZERO eigenvalue, as this state does not evolve with time. $\hat{H}\|\psi\rangle = E_0\|\psi\rangle = 0$ |
|---|

e.g for $\hat{c}=\hat{a}$    the pointer state that system relaxes to is: $|0\rangle$

$$a|0\rangle = 0|0\rangle$$

Next consider: $\hat{c}=(\mu a + \nu\, a^+)$ $\wedge$ $\mu^2-\nu^2=1$ e.g $\mu=\cosh(r)$, $\nu=\sinh(r)$

The eigenstate is: $|\xi\rangle$ a squeezed state.

$\hat{X}_1 \equiv \hat{a}+\hat{a}^+$ Quadratures
$\hat{X}_2 \equiv \hat{a}-\hat{a}^+$

Therefore the system relaxes to a $|\xi_0\rangle$ squeezed vacuum

$$\langle\Delta\hat{X}_1^2\rangle = \frac{1}{2}e^{-2r}$$
$$\langle\Delta\hat{X}_2^2\rangle = \frac{1}{2}e^{+2r}$$

Next consider: $\hat{c}=(a-\alpha)(a-\beta)$ for $\alpha=-\beta$ this yields

the eigenstate $(|\alpha\rangle + |-\alpha\rangle) = |\psi\rangle$ (*) and $\hat{c}|\psi\rangle = 0$

$\Rightarrow$ the system relaxes to a Schrödinger cat state (*)

other schemes exist for entangled states (e.g $(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B)\frac{1}{\sqrt{2}}$

or for a 3level system

$\hat{c}_p|D\rangle = 0$
dark state

Reference: "Introduction to Quantum Noise, Measurements, & Amplification". Clerk, Rev. Mod. Phys.

## Spectral Densities in Quantum Mechanics

Let us define a quantum mechanical equivalent of the spectral density:

$$S_{xx}[\omega] = \int_{-\infty}^{\infty} dt\, e^{i\omega t} \langle \hat{x}(t)\hat{x}(0)\rangle$$

where $\hat{x}$ is a Heisenberg operator :

Introductory example: the Harmonic oscillator:

$$\hat{H} = \left(m\omega^2\hat{x}^2 + \hat{p}^2\frac{1}{2m}\right) \wedge \quad [\hat{x},\hat{p}] = i\hbar \quad \Rightarrow \frac{d}{dt}\hat{x} = -\frac{i}{\hbar}[\hat{H},\hat{x}]$$

Thus the position autocorrelation $G_{xx} \equiv \langle \hat{x}(t), x(0)\rangle$ is given by

$\hat{x}(t) = \hat{x}(0)\cos(\omega t) + \frac{1}{m\omega}\hat{p}(0).\sin(\omega t)$ is a solution of the Schrödinger equation. Thus:

$$\langle\hat{x}(t)\hat{x}(0)\rangle = \langle\hat{x}(0)\hat{x}(0)\rangle\cdot\cos(\omega t) + \frac{1}{m\omega}\langle\hat{p}(0)\hat{x}(0)\rangle\sin(\omega t)$$

Classically the right term vanishes since in thermal equilibrium $\hat{x}, \hat{v} =$ $x(t)$ and $v(t) = p(t)/m$ are independent processes (see our earlier treatment of Brownian motion where $(\langle x^2\rangle \neq 0; \langle v\rangle^2 = \frac{k_BT}{m}; \langle x(t)v(t)\rangle =$

In the QUANTUM case however the SYMMETRIZED spectrum vanishes but not the non-symmetrized spectrum. Let us analyze:

$$\langle[\hat{x}(0),\hat{p}(0)]\rangle = \langle\hat{x}(0)\hat{p}(0)\rangle - \langle\hat{p}(0)\hat{x}(0)\rangle = \langle\hat{x}(0)\hat{p}(0) - \hat{p}(0)\hat{x}(0)\rangle = i\hbar$$

and: $\left.\begin{array}{l}\langle\hat{p}(0)\hat{x}(0)\rangle = -\frac{i\hbar}{2} \\ \langle\hat{x}(0)\hat{p}(0)\rangle = +i\hbar/2\end{array}\right\}$ using $\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^{+}) \wedge \hat{p}$ $\quad [a,a^{+}] = 1$

Thus even since $\hat{x}$ is an hermetian operator (with therefore REAL eigenvalues) $G_{xx}$ is COMPLEX: since the cross correlator $\langle\hat{x}(0)\hat{p}(0)\rangle$ is COMPLEX:

$$G_{xx} = \langle\hat{x}(0)^2\rangle\cos(\omega t) + \frac{1}{m\omega}\left(\mp\frac{i\hbar}{2}\right)\sin(\omega t)$$

$$G_{xx} = \left(\frac{\hbar}{2m\omega}\right)\left[\,n(\hbar\omega)\,e^{+i\omega t} + [n(\hbar\omega)+1]\,e^{-i\omega t}\right]$$

where $n(\hbar\omega)$ is the Bose Einstein factor of the oscillator (when being in a thermal state, i.e. $\langle\hat{x}(0)^2\rangle = 2x_{ZPF}^2\cdot\{\bar{n}(\hbar\omega)+1\}$

1

## Spectral Densities in Quantum Mechanics
## (Quantum Noise)

Since the correlator is <u>not</u> symmetric the spectral density is also not symmetric in frequency

$$S_{xx}(\omega) = 2\pi\, x_{ZPF}^2 \left\{ n(\hbar\omega)\,\delta(\omega+\Omega) + [n(\hbar\omega)+1]\,\delta(\omega-\Omega) \right\}$$

Hence the quantum spectral density is frequency assymetric.

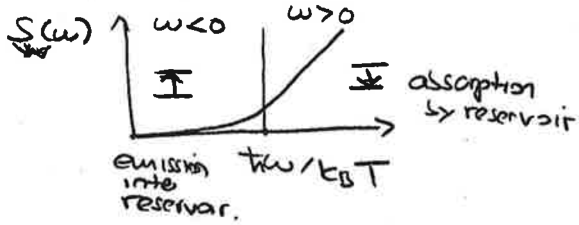
<u>Physical picture:</u>

$\omega > 0$ (positive frequencies): stimulated emission into the oscillator

$\omega < 0$ (negative frequencies) emission of energy by the oscillator

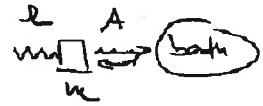
<u>Ref</u>: → "Introduction to Quantum Noise" Clerk et al Rev. Mod. Phys.

Example: Voltage fluctuations accross a cold resistor $S_{VV}(\omega)$




<u>Note</u>: in last homework a quantum spectrum analyser was shown (a TLS coupled to a drive field)

A further example of a QUANTUM NOISE SPECTROMETER is a harmonic oscillator of mass $M$, frequency $\Omega$ position $x$ that is coupled to a force: $\hat{H}_c = A\hat{x}\hat{F} = A\,x_{ZPF}(a+a^\dagger)\hat{F}$




This allows in principle to measure $S_{FF}(\omega)$

using the quantum Master equation we can show that

$$\frac{d\langle E\rangle}{dt} = \frac{d}{dt}\mathrm{Tr}(\rho\,\hbar\omega a^\dagger a) = P - \gamma\langle E\rangle$$

$\gamma$: energy relaxation

$P$: heating of the oscillator by the bath

which are given by the symmetric and assymmetric in frequency part of $S_{FF}(\omega)$.

$$\boxed{\begin{aligned} \gamma &= \frac{A^2 x_{ZPF}^2}{\hbar^2}\left\{ S_{FF}(\Omega) - S_{FF}(-\Omega) \right\} \\ P &= \frac{A^2}{4M}\left\{ S_{FF}(\Omega) + S_{FF}(-\Omega) \right\} \hat{=} \frac{A^2}{2M}\overline{S}_{FF}(\Omega) \end{aligned}}$$

note: related to the FDT by Callen & Welton

## Spectral Densities

relation to the Fluctuation dissipation theorem:

- Use Fermi's Golden rule to calculate transition rates

$$\Gamma_\uparrow = |\langle n|A\hat{F}\hat{x}|n+1\rangle|^2 = \frac{A^2}{\hbar^2} S_{FF}[-\Omega] \qquad \text{(Fermi's Golden rule)}$$

$$\Gamma_\downarrow = |\langle n|A\hat{F}\hat{x}|n-1\rangle|^2 = \frac{A^2}{\hbar^2} S_{FF}[+\Omega]$$

In thermal equilibrium we have:

$$\frac{\Gamma_\uparrow}{\Gamma_\downarrow} = e^{-\hbar\Omega/k_BT} \qquad \text{thus:} \quad S_{FF}(+\omega) = e^{\hbar\omega/k_BT} S_{FF}(-\omega)$$

thus:

$$\bar{S}_{FF}(\Omega) = \frac{1}{2}\left(S_{FF}(\Omega) + S_{FF}(-\Omega)\right)$$

$$= \frac{1}{2}\coth(\beta\hbar\Omega/2)\left(S_{FF}(\Omega) - S_{FF}(-\Omega)\right)$$

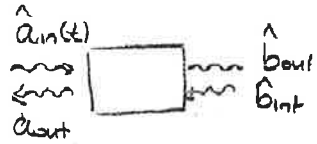
$$\Leftrightarrow \quad \boxed{\bar{S}_{FF}(\Omega) = \coth\left(\frac{\hbar\Omega}{2k_BT}\right)\frac{\hbar\Omega m}{A^2}\gamma[\Omega]}$$

Fluctuation dissipation theorem by Callen & Welton 1951 (valid also for $T \to 0$)

Lecture 13

<u>Application of Open quantum systems to amplification</u> "Haus–Caves Theory"

2nd Treatment: | Caves "Quantum limits on noise in linear amplifiers" Phys Rev D (1982)




Assume we are amplifying an Electrical field of a signal

$$E(t) \propto a e^{-i\omega t} - a^* e^{+i\omega t}$$ (CLASSICALLY) i.e. $\propto 2\sin(\omega t)\,\mathrm{Im}\{a$

and quantum mechanically

$$\hat{E}(t) \propto \hat{a} e^{-i\omega t} - \hat{a}^+ e^{+i\omega t}$$ (QUANTUM OPERATOR FOR $\hat{E}$)

Let $\hat{a}$ be input and $\hat{b}$ be output operators.

The symmetrized noise in the quantity is:

$$\boxed{\langle \Delta a^2 \rangle \equiv \langle \tfrac{1}{2}(a a^+ + a^+ a) - \langle \hat{a} \rangle^2 \rangle \mathrel{\hat{=}} \langle \tfrac{1}{2}\{a, a^+\} - \langle \hat{a} \rangle^2 \rangle}$$

To derive the quantum limit the commutators satisfy

$[a, a^+] = 1 \quad \wedge \quad [b, b^+] = 1$

Assume that $\hat{b} = \sqrt{G}\,\hat{a} \quad \wedge \quad \hat{b}^+ = \sqrt{G}\,\hat{a}^+$ G: gain

We need to write:

$$\hat{b} = \sqrt{G}\, a + \hat{F} \qquad \hat{b}^+ = \sqrt{G}\, a^+ + \hat{F}^+$$

$F$ represents the Noise and is not correlated with input $\langle \hat{F}\hat{a}^+ \rangle = 0 = \langle \hat{F}\hat{a} \rangle$

Stipulating $[\hat{b}\hat{b}^+] = 1$ yields: (and $\langle \hat{F} \rangle = \langle \hat{F}^+ \rangle = 0$)

$$\boxed{[F, \hat{F}] = 1 - G}$$

This implies that Noise is also present in the output mode. Specifically:

$$\langle \Delta b^2 \rangle \equiv \langle \tfrac{1}{2}(\hat{b}\hat{b}^+ + \hat{b}\hat{b}^+) - \langle b \rangle^2 \rangle = (\sqrt{G})^2 \langle \Delta a^2 \rangle + \langle \tfrac{1}{2}(\hat{F}\hat{F}^+ + \hat{F}^+\hat{F}) - \underbrace{\langle F \rangle^2}_{0} \rangle$$

0 for white noise.

$$= G\langle \Delta a^2 \rangle + \frac{1}{2}\langle \{\hat{F}, \hat{F}^+\} \rangle$$

$$\geq G\langle \Delta a^2 \rangle + \frac{1}{2}\left|\langle [\hat{F}, \hat{F}^+] \rangle\right|$$ Large gain limit G:

$$\boxed{\langle \Delta b^2 \rangle \geq G\langle \Delta a^2 \rangle + \frac{1}{2}(G-1)}$$ or $$\boxed{\frac{\langle \Delta b^2 \rangle}{G} \geqq (\Delta a)^2 + \frac{1}{2}}$$ NOISE REFERRED TO INPUT

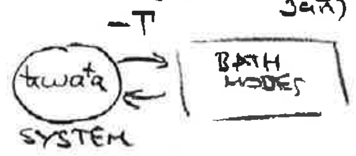


Simplest form ("=") [illegible] HALF QUANTUM OF NOISE

## Amplifiers and Measurements and Noise limits

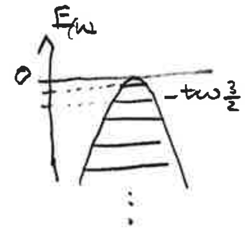
Minimum added noise by an amplifier. Let us consider a simple amplifier model:

(negative damping ie gain)




We need a medium with GAIN

Two Models are possible. First consider negative temperature bath: (Glauber)




$$E_n = -(n+\tfrac{1}{2})\hbar\omega$$

Maintained at a negative ($T<0$) Temperature T.

$$\rho_{nn} = p_n = \frac{e^{-\hbar\omega/k_BT}}{1-e^{-\hbar\omega/k_BT}}$$

ie for T small $p_{11} > p_{22} > p_{33}$.

Since raising an lowering operators now have opposite effect (lowering increases the number of quala) the roles are reversed. Thus

$$\left| \hat{H}_{Bath} = (\hbar\Omega_k)(n+\tfrac{1}{2})(h_k h_k^{+}) \right| \qquad \text{ie } \hat{n} = h_k h_k^{+} \quad (\text{not } \hat{a}^{+}\hat{a} \text{ in this reversed case})$$

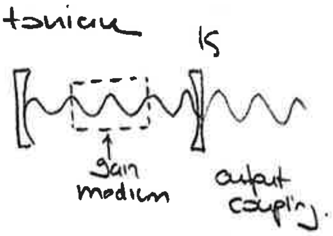
For a NEGATIVE TEMP State we have:

$$\left| \begin{array}{l} \langle h_k^{+}(t) h_k(t')\rangle = (\bar{n}+1)\,\delta(t-t') \\ \langle h_k(t) h_k^{+}(t')\rangle = (\bar{n})\,\delta(t-t') \end{array} \right|$$

This is a simple model for GAIN. (e.g. laser action)

Quantum Stochastics Equations:

$$\left| \begin{array}{l} \hat{H}_{gain} = i\hbar \int \gamma(\omega)\,[\,h(\omega)\,a^{+} - a\,h^{+}(\omega)\,]\,d\omega \\ \hat{H}_{loss} = i\hbar \int \kappa(\omega)\,[\,b(\omega)\,a^{+} - a\,b^{+}(\omega)\,]\,d\omega \end{array} \right| \qquad \text{Interaction Hamiltonian}$$




$\Rightarrow$

$$d_t a = +\underbrace{\frac{\gamma}{2}a}_{\text{gain}} + \underbrace{\sqrt{\gamma}\,h_n(t)}_{\text{noise term}} - \underbrace{\frac{\kappa}{2}a}_{\text{loss}} - \underbrace{\sqrt{\kappa}\,b_{in}(t)}_{\text{noise term}}$$

Solve formally:
$$a(t) = a(0)\,e^{-\frac{(\kappa-\gamma)t}{2}} + \int_0^t dt'\, e^{-\frac{(\kappa-\gamma)}{2}(t-t')/2}\,[\sqrt{\gamma}\,\underbrace{h(t')}_{\approx h(t)} - \sqrt{\kappa}\,\underbrace{b_{in}(t')}_{\approx b(t)}]$$

approximate $h(t)$ fluctuates fast compared to $(\kappa-\gamma)^{-1}$

## Amplifiers, Measurements and Noise limit

simplify to $\quad a(t) \approx \frac{2}{(\kappa-\gamma)} \left\{ \sqrt{\gamma}\, h_{in}(t) - \sqrt{\kappa}\, b_{in}(t) \right\}$

The input-output relation is:

$$b_{out}(t) = b_{in}(t) + \sqrt{\kappa}\, a(t)$$

Thus

$$b_{out}(t) = \underbrace{\left(\frac{\gamma+\kappa}{\gamma-\kappa}\right)}_{>1 \text{ i.e amplified input}} b_{in}(t) - \underbrace{\frac{2\sqrt{\gamma}\sqrt{\kappa}}{\gamma-\kappa}\, h_{in}(t)}_{\text{noise added in the amplification process}}$$

GAIN: $G \equiv \left(\frac{\gamma+\kappa}{\gamma-\kappa}\right)$

Added NOISE at the AMPLIFIER:

Fourier transform and solve eqs. $\quad \left[ \frac{d}{dt} \to i\omega \quad a(t) \to a(\omega) \right]$

$$a(\omega) = \frac{\sqrt{\gamma}\, h_{in}(\omega) - \sqrt{\kappa}\, b_{in}(\omega)}{i\omega + (\kappa-\gamma)/2} \qquad \text{from QLE.}$$

Thus

$$b_{out}(\omega) = b_{in}(\omega) + \sqrt{\kappa}\, a(\omega)$$

$$\boxed{b_{out}(\omega) = b_{in}(\omega) \underbrace{\frac{-2i\omega + (\gamma+\kappa)}{-2i\omega + (\gamma-\kappa)}}_{G(\omega) \text{ frequency dependent gain}} - \underbrace{h_{in}(\omega) \cdot \frac{2\sqrt{\kappa}\sqrt{\gamma}}{-2i\omega + (\gamma-\kappa)}}_{\text{added Noise}}}$$

- Gain is maximum at $\omega = 0$ and $G = 1$ for $\omega \gg (\gamma+\kappa)$

Added noise is given by the mean square (r.m.s) fluctuations in the output quanta per UNIT BANDWIDTH i.e this is a SPECTRAL DENSITY OF FLUCTUATIONS

$$\boxed{\begin{aligned} &\delta(\omega-\omega') \cdot S_{b_{in}b_{in}}(\omega) \equiv \hbar\omega \left\langle \left(b_{in}^{+}(\omega) - \langle b_{in}^{+}(\omega)\rangle\right)\left( b_{in}(\omega') - \langle b_{in}(\omega')\rangle\right)\right\rangle = \\ &\delta(\omega-\omega') \cdot S_{b_{in}b_{in}}(\omega) \triangleq \hbar\omega \langle \delta^{+} b_{in}(\omega)\, \delta b_{in}(\omega')\rangle \qquad \text{where } \delta\hat{b} = \hat{b}(\omega) - \langle \hat{b}(\omega)\rangle \end{aligned}}$$

The units are quanta-squared per Hertz $(1/s^{-1})$ i.e s (seconds).

From the definition it follows that

$$S_{\hat{b}_{out},\hat{b}_{out}}(\omega) = S_{\hat{b}_{in},\hat{b}_{in}}(\omega) \cdot |G(\omega)|^2 + S_{\hat{h}_{in},\hat{h}_{in}} \left( \underbrace{|G(\omega)|^2 - 1}_{\text{comes for re-expression } \frac{4\kappa\gamma}{|2i\omega + (\gamma-\kappa)|^2}} \right)$$

# Amplifier & Noise limits

The added noise is given by:

$$A(\omega) = S_{\hat{b}_{in},\hat{h}_{in}}(\omega)\,(1 - |G(\omega)|^2) \underset{G\gg 1}{=} S_{\hat{h}_{in}\hat{h}_{in}}(\omega)$$

In the case of a zero tempeature bath $S_{\hat{h}_{in}\hat{h}_{in}}(\omega)\cdot\delta(\omega-\omega') = \langle \delta\hat{h}^{+}_{in}(\omega)\,\delta\hat{h}_{in}\ldots \times\hbar\omega$

becomes

$$S_{\hat{h}_{in}\hat{h}_{in}} = \hbar\omega \times 1$$

since i.e. $\langle \hat{h}^{+}_{in}(t)\,\hat{h}_{in}(t')\rangle = (n+1)\,\delta(t-t')$

or: $\langle h^{+}_{in}(\omega)\,h_{in}(\omega')\rangle = (n+1)\,\delta(\omega-\omega')$

Thus

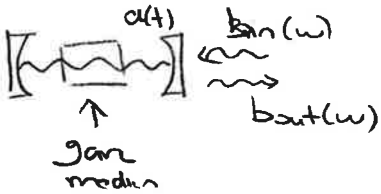
$$\boxed{A(\omega)_{T=0} = \hbar\omega\cdot(1 - |G(\omega)|^2)}$$

Hence we have 1 quanta of added Noise per unit bandwidth (1 Hz).

$$|A(\omega)|^2_{G\gg 1} = \hbar\omega \qquad \text{for } T=0$$

- Thus even for an absolute zero temperature bath the amplifier will add noise unless that $G=1$ (no gain)

## SIGNAL TO NOISE ANALYSIS




Signal input power

$$S_{in} \equiv \hbar\omega\,|\langle \hat{b}_{in}(\omega)\rangle|^2 \qquad \left(\text{units: } \frac{s^{-1}}{s^{-2}} = s\right)$$

Note: This expression has units of $[\langle \hat{b}_{in}(\omega)\rangle]^2 = \left(\frac{1}{s}\right)^2 \times s$ since $[\langle b^{+}_{in}(t)\,b_{in}(t)\rangle] = 1/\text{second}$ (i.e. a Flux of photons).

Hence the signal has same units as the noise. We define

$$SNR \equiv \left(\frac{S_{out}}{N}\right) \qquad \text{where} \qquad S_{out} = S_{in}\,|G(\omega)|^2$$

Thus:

$$\left(\frac{S}{N}\right)^{-1}_{out} = \left(\frac{S}{N}\right)^{-1}_{in} + \left(-|G(\omega)|^2 + 1\right)\frac{N}{S_{in}} \qquad \text{with } N = S_{h_{in},h_{in}}(\omega)$$

Thus, $G\gg 1$

$$\left(\frac{S}{N}\right)^{-1}_{out} = \left(\frac{S}{N}\right)^{-1}_{in} + \underbrace{\frac{\hbar\omega\,(\bar{n}+1)}{S_{in}}}_{\text{always positive}}$$

The signal to noise always reduces

best case $\bar{n}=0$

$$\boxed{\left(\frac{S}{N}\right)^{-1}_{out} = \left(\frac{S}{N}\right)^{-1}_{in} + \hbar\omega\,\frac{1}{S_{in}}}$$

Noise Temperature

$$\frac{\hbar\omega}{e^{\hbar\omega/k_B T_{noise}} - 1} \equiv \left(1 - \frac{1}{G^2}\right) \cdot S_{u_{in} u_{in}}$$

Idea: express amplifier noise as if the noise was due to extra noise (thermal noise) by increasing Temperature of the input mode